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Reader comments and additional notes on the book *The Language of Mathematics: Utilizing Math in Practice* by Robert Laurence Baber

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Note: Wherever “the book”, “this book”, etc. appears
in the comments and additional notes below,
“book” refers to and means the book

The Language of Mathematics: Utilizing Math in Practice.

1. The logical equivalence function and its infix symbol ≡

In Boolean expressions the infix function symbol ≡ is sometimes used instead of = or ⇔. Applied to the domain $\mathbb{B}\times\mathbb{B}$ these three functions all have the same values. Their binding orders differ: ≡ is normally defined to be either in binding order 9 (together with ⇒, ⇐, and ⇔) in TABLE 3.4.2-1 on page 45 of the book or in a new binding order 10. Writing ≡ instead of = reduces the need for parentheses in many Boolean expressions. Writing ≡ instead of ⇔ better expresses visually the fact that an equivalence relation is meant.

2. Terminology for derivatives

One often refers to "the derivative of f(x) with respect to x" as in the first sentence at the top of page 113 in Section 4.5 of the book. This type of phrase can be seen in several forms in the literature, for example:

1. the derivative of f(x) with respect to x
2. the derivative of the function f(x) with respect to x
3. the derivative of the expression f(x) with respect to x
4. the derivative of an expression with respect to x
5. the derivative of the function f with respect to x
6. the derivative of the function f with respect to its argument
7. the derivative of the function f with respect to its first argument
8. the derivative of the function f with respect to its second argument
9. the derivative of the function f with respect to its … argument

These phrases occur in the interpretation in English of mathematical expressions and parts of mathematical models. As such, they are subject to the "normal" ambiguities of English.

Strictly speaking, f(x) is not a function, it is an expression referring to the value of the function f for an argument value that is the value of the variable x. Therefore, phrase 3 above is generally preferred over phrase 2. The form of phrase 2 emphasizes that the function f is of primary interest and includes the information that x is the name of the variable that is both the argument of the function and the variable with respect to which the derivative is being taken. This is a good example of hidden information, ambiguity, and emphasis, all of which are common in natural language text.

In phrase 1 above, the part "of f(x)" implies "of expression f(x)", so phrase 1 can be viewed as an abbreviated form of phrase 3.

Phrase 5 above is reasonable only if it is clear from the context that x is the argument of the function f. If this is not clear from the context, then this phrase is ambiguous and unacceptable.

When the derivative is a partial derivative, the word "partial" should be included in the phrases above.

In summary, the preferred notational descriptions for a derivative are 3, 4, and 7 through 9. If the function f has only one argument, then phrase 6 is also among the preferred descriptions. Phrase 1, an abbreviated form of phrase 3, is also unambiguous and fully acceptable.

3. Comments from Prof. Dr.-Ing. Adolf J. Schwab received 2011 September 29

Dear Bob,

thank you very much for the complimentary copy of your book "Language of Mathematics". This is a monument! I do agree that your book can inspire many teachers of mathematics in conveying a better understanding and appreciation.

Unfortunately, I am afraid that you may face some gusty winds from puristic math professors regarding mathematical rigor. For example, in Chapter 4.5 Calculus you do not distinguish between incremental quantities Δx and infinitesimal quantities dx. You state that "Each letter d indicates the "difference" or "differential" of the following value......" However, there is a distinct difference between Δx and dx. You can see Δx and represent it in a graph as a very small but finite length, but you cannot see dx. Δx becomes dx for Δx striving towards zero! dx is the result of taking a limit! Therefore, your equation 4.5-4, showing dx → 0 according to usual nomenclature may provoke criticism (see correct writing handwritten at the end of this text). For mathematical rigor, you always have to start with an incremental, finite Δx, then take the limit for Δx going to zero. Although dx is infinitely small, a ratio of d "operand"/dx remains finite.

Further, an indefinite integral must in its general definition always include a Constant C. This constant C is essential and cannot only show up in the accompanying text.

Finally, in your Chapter on Probability Theory I am missing the concept of the density function etc.

Whatsoever, I consider your book a highly valuable contribution and challenge to improving the didactics of mathematics. It provides a lot of food for thought of how teaching math could be done better, congratulations!!!!!!!!!!!!!!

Hoping that you are doing well I remain with warm regards

Adolf

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Equ 4.5-4 must read

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

In taking the limit, the "difference ratio" becomes the "differential" ratio, i.e. the slope in x.

3a. The author's response to Prof. Dr.-Ing. Adolf J. Schwab's comments

I thank Prof. Dr.-Ing. Adolf J. Schwab very much for his encouraging statements about this book.

Second paragraph in Prof. Schwab's letter

The text in Prof. Schwab's paragraph on dx and Δx is very similar to the material from which I learned derivatives many years ago, so I agree with these comments. This paragraph raises several didactically interesting and useful points and issues in nomenclature, mathematical terminology in English, and mathematical notation. These points and issues deserve elaboration. Prof. Schwab's second paragraph raises some of them explicitly and others, indirectly and implicitly.

In preparation for my responses to this paragraph, I refer the reader to Section 6.10 of the book, pages 193 to 195. There it is pointed out that mathematical text is typically written in a mixture of (1) purely mathematical expressions, (2) commentary in English, and (3) the interpretation in English of mathematical expressions and parts thereof. Section 6.10 further points out that the reader should distinguish clearly between these three components of mathematical text. I will refer to these three categories of components of mathematical texts in some of my responses to the particular points and issues below.

defining a derivative: In Section 4.5, on page 113 of the book, the right side of equation [4.5-4] gives one common form for the definition of the derivative of the expression f(x) with respect to the variable x. Another common form for the definition of this derivative follows from equation [4.5-1]:

$$\frac{df(x)}{dx} = \lim_{x1 \to x} \; (f(x1)-f(x)) \,/\, (x1-x)$$

Both forms are found in the mathematical literature, as are variations of them, for example, the above form with x instead of x1 and x0 instead of x:

$$\frac{df(x0)}{dx0} = \lim_{x \to x0} \; (f(x)-f(x0)) \,/\, (x-x0)$$

These three forms (equation [4.5-4] in the book and the two equations above) are all equivalent ways of defining a derivative. In the case of the last equation above, x0 is the free variable, instead of x in the other two, so the derivative is a function of the function f and the value of the variable x0 instead of x.

Notice that neither the name "dx" nor the name "Δx" appears in either of the two definitions above.

The definitions themselves are to the right of the = symbol in each expression above. The term to the left of the = symbol in each expression is a notational form for the derivative of an expression with respect to a variable. That notational form should be construed as a composition of a derivative symbol and the arguments of the derivative function (see the paragraphs on **notational forms for derivatives** below).

Notice also that the variable dx in equation [4.5-4] in the book, Δx in Prof. Schwab's comments, x1 in the first equation above, and x in the second equation above are quantified, bound, "dummy", "running" variables. This fact can probably be seen most clearly in equation [4.5-5] in the book, in which dx is quantified in the innermost quantified expression. As pointed out in Section 3.4.8 at the bottom of page 67 of the book, the name of such a variable can be changed without affecting the value of the expression in which it appears, provided that the new name is not the same as any other name appearing in the expression.

Therefore, the variable names dx in the book, Δx in Prof. Schwab's comments, x1 above, and x in the second equation above can be changed to any other name not already appearing in the expression in question. Whether or not the letter "d" or "Δ" appears in the name of any of these variables is irrelevant and of no consequence; mathematically, those variables could just as well have been named p, y, z, abc, or whatever. The names dx and Δx are suggestive of meanings useful to human readers, but those names are of no mathematical significance. Only the properties of the values of the variables are of mathematical significance.

It is useful to examine equation [4.5-5] in the book and identify the names of the free (unbound) variables and functions. In addition to the set $\mathbb{R}$, which can be viewed as a constant as is the value 0, and the standard mathematical functions $\wedge$, $\vee$, $\in$, $>$, $<$, $|\ldots|$, $-$ and /, the names of the free variables and functions are f, x, and g. Therefore, if g(x) exists, (that is, if any g(x) satisfies equation [4.5-5]), g(x) can depend only on f and x. This is a general result: the derivative of an expression with respect to a variable depends only on the expression and (the value of) that variable. This has implications for notational forms for derivatives.

notational forms for derivatives: Some of the many notational forms for derivatives appearing in the literature are listed below. As stated in the book in Section 4.5 in expression [4.5-3] and in the text at the bottom of page 112, the most commonly used notational forms are the first two listed below:

1. $\dfrac{df(x)}{dx}$
2. $df(x)/dx$
3. $f'(x)$
4. $\dot{y}(x)$
5. $D_x f$
6. $D_x f(x)$
7. df/dx
8. $\dfrac{d}{dx} f(x)$

The most common forms 1 and 2 as well as forms 7 and 8 above have a disadvantage: the sequence of letters "dx" is visually present, suggesting the variable name dx and misleading some into thinking that some residual of this variable still has an effect on the result. An advantage of these forms is that they reflect and remind the reader of the form of the definition in equation [4.5-4] on page 113 of the book. However, these forms would perhaps be better read as

$$\frac{d}{d}(f(x), x)$$

or (d/d)(f(x), x) to remind the reader that the derivative of an expression f(x) with respect to the variable x is a function of the expression f(x) and the value of the variable x, and that the variable dx plays no role in the result itself (only in the way that result is determined). The author has never seen this notational form in the mathematical literature.

In any case, the reader is well advised to view the "d" in the denominator of forms 1, 2, 7, and 8 as a part of a derivative notation "d/d" and not as part of a variable name dx. The "x" in the denominators of these forms should be viewed as the variable name x itself, completely divorced from the "d".

Form 6 above probably expresses most clearly the fact that the derivative of an expression f(x) with respect to x is a function of that expression and the variable named x. Despite this advantage, form 6 is not in widespread use. The similar notational form D(f(x), x) would have the same advantage, but the author has never seen this form in the literature.

dx vs. Δx as a variable name: In Section 4.5 of the book, dx is used as the name of the variable that is named Δx in Prof. Schwab's letter. As concluded in the paragraphs on **defining a derivative** beginning on page 4 above, the name of this variable, which is a quantified variable in the expressions in which it appears, is irrelevant and of no consequence outside the expression defining the derivative in question.

Prof. Schwab distinguishes between the meanings of the names dx and Δx. The text in the book does not make this distinction. The concept of dx as described in his letter is absent from the book, see the paragraphs on "incremental" and "infinitesimal" below. In summary, if readers find this distinction useful and helpful in learning this material, fine. Mathematically it is not necessary, however, to make or to consider this distinction, and readers who find it confusing are best advised to disregard it. The notion of a variable or an expression approaching zero as a limit is sufficient (and necessary).

incremental vs. infinitesimal, difference vs. differential, infinitely small: These terms appear not only in Prof. Schwab's letter, but also in various texts on the subjects of calculus, derivatives and integration.

I will consider individually these terms and their definitions given in three of my favorite dictionaries:

- *Mathematics Dictionary, Multilingual Edition* by James and James, D. Van Nostrand Company, Inc., 1968
- *Webster's New World College Dictionary*, Third Edition, Victoria Neufeldt, Editor in Chief and David B. Guralnik, Editor in Chief Emeritus, Macmillan, 1995
- *Webster's New Collegiate Dictionary*, G. & C. Merriam Co., 1949

Where a mathematically oriented definition is given in a general dictionary, I will concentrate on that definition and pay less or no attention to other definitions clearly intended for a broader use.

As a general warning in advance, one should read critically the definitions of terms used in a mathematical context when such definitions are given in dictionaries aimed at a general readership.

incremental, difference: The terms "incremental" and "difference" are straightforward and their meanings are comparatively clear and unproblematic. For "incremental", dictionary definitions mention a difference between two numbers or increases and decreases in the value of some variable. For "incremental" some dictionaries include the statement that the increase or decrease is "usually small". None of these definitions mention a limit.

infinitesimal: In the mathematical sense, all three dictionaries mentioned above give the definition "a variable which approaches zero as a limit" or some minor rewording thereof. The variable dx in the book and the variable Δx in Prof. Schwab's letter both fulfill this definition, so they can be called infinitesimals. The two general dictionaries listed above also give such definitions as "too small to be measured", "infinitely small", "immeasurably or incalculably small", "very minute". "Too small to be measured" is clear, but is of consequence only in practical, but not in theoretical work. It is not at all clear to me what "incalculably small" is intended to mean; one can calculate with arbitrarily small numbers. The term "very minute" is very vague in meaning; while it has its place in normal natural language, it does not belong in a mathematically precise text.

Notice that the primary difference between the definition of "incremental" and "infinitesmal" is that an infinitesmal approaches zero as a limit, while the term "incremental" does not imply or necessarily involve any variable approaching any limit.

infinitely small: Linguistically, the phrase "infinitely small" is odd. I found no definition of this phrase in my dictionaries, so examined "infinitely" in order to associate its definitions with the relatively clear adjective "small". Among the definitions of "infinitely" or its adjective form "infinite" are: "becoming large beyond any fixed bound", "indefinitely large", "very great", "vast", "immense", "immeasurable", "indefinitely large", etc. With the exception of one interpretation of "immeasurable", all of these definitions clearly indicate something large or great, which clearly contradicts the idea of "small". Therefore, the phrase "infinitely small" is, in itself, self-contradictory and therefore meaningless, particularly in a mathematical context.

The term "immeasurable" could mean either immeasurably large or immeasurably small. The meaning "immeasurably large" belongs to the group already discussed in the paragraph above. The meaning "immeasurably small" is the same as "too small to be measured" or "incalculably small", both of which were discussed in the first paragraph on **infinitesimal** above.

Probably "infinitely small" is intended to mean "very small" and the word "infinitely" is used instead of "very" or "extremely".

Alternatively, "infinitely small" could mean "infinitesimally small" or, more simply, "infinitesimal". Then a variable approaching zero as a limit is meant, see the paragraphs on **infinitesimal** above.

differential: Probably the best definition of the word "differential" in the context of calculus is "the derivative of a function multiplied by a small increment of the independent variable". Notice the word "small". In the common mathematical use of this word in calculus it applies to an expression in which the small increment is understood to approach zero as a limit. Typical examples are the subexpression

$$f(x)\ dx$$

in expression [4.5-7] on page 114 and other similar subexpressions on following pages of Section 4.5 of the book.

In an alternative definition of the integral [4.5-7] on page 114 of the book the integral is defined or interpreted to be the area under the graph of f(x) (cf. the graph at the bottom of page 115 of the book). This area is approximated by rectangles of height f(x) and width dx, i.e. each with area f(x)dx. As dx approaches zero as a limit the sum of these terms (areas of the rectangles) approaches the area under the graph as a limit.

If f(x) is bounded, then every f(x)dx also approaches zero as a limit as dx approaches zero as a limit, so both dx and f(x)dx satisfy the definitions of the terms "differential" and "infinitesimal".

"You can see Δx and represent it in a graph as a very small but finite length, but you cannot see dx": I would express this differently: One can (at least in principle) see a non-zero length on a graph, but one cannot see the length of the limit, which is zero. Also, instead of the word "finite", the word "non-zero" would be better. See also my discussion of **dx vs. Δx as a variable name** on page 6 above.

"Each letter "d" indicates the "difference" or "differential" of the following value": This quotation from the book (Section 4.5, last sentence on page 112) falls in category (2) on page 4 above. This sentence is intended to motivate the choice of the name of the variable dx for the difference between two values of x, and to motivate the reference df(x) to the difference between the values of the expression f(x) for two different values of x. Notice that the phrase "df(x)" is not itself a mathematical expression. If it were, it would be the value of a function df for the argument value x.

Notice that both of the differences mentioned in the paragraph above are in a context in which they approach zero as a limit, so they also satisfy the definitions of an infinitesimal and of a differential.

See the comments on **notational forms for derivatives** on page 5 above, especially the suggestion to view the notation df(x)/dx as

$$\frac{d}{d}(f(x), x)$$

separating distinctly the two occurrences of "d" from "f(x)" and "x".

Third paragraph in Prof. Schwab's letter

the constant of integration: Prof. Schwab's comment is correct. In Section 4.5, on pages 114 and 115 of the book, I made two points regarding two functions f and g:

1. If f is the derivative of g, then g is the indefinite integral of f, and vice versa.
2. Under these conditions, f is uniquely determined by g, but g is not uniquely determined by f. Any function g1 that differs from g only by a constant is also an indefinite integral of f.

For reasons of clarity, especially for readers encountering this topic for the first time, I chose to complete making point 1 before starting on point 2. It would have been more precise to replace "g is the integral of f" by "g is an integral of f", but this would have implicitly introduced point 2 before point 1 was complete.

In any case, equation [4.5-8] on page 114 of the book is correct in that it gives one possible g(x) for a given f(x). It is not, however, the only function of x that satisfies equation [4.5-8]. If g(x) satisfies equation [4.5-8], then so does g(x)+c, where c is any constant, as is pointed out in the first paragraph on page 115 of the book. In fact, c may be any expression not dependent on the value of the variable x.

Fourth paragraph in Prof. Schwab's letter

probability density function: In Section 4.6.1 of the book, at the end of the last complete paragraph on page 119, is the sentence "Otherwise, and particularly if the sample space S is infinite and not countable, other approaches to defining the set A and the probability function p must be employed". In the case of a sample space involving one random variable with values in $\mathbb{R}$, a probability *distribution* function Pdist is defined such that Pdist(x) is the probability that the value of the random variable is less than or equal to x. The probability *density* function is the derivative of Pdist(x) with respect to x. The set A of subsets of S is defined in an appropriate way (often as Borel sets). If the random variable takes on values in some set other than $\mathbb{R}$, then the set A and the probability distribution and density functions are typically defined in a corresponding way. These aspects of probability theory go beyond the scope of this book, as do many aspects of statistics, so these topics were not included. The interested reader is referred to the many books on probability theory and statistics.

In closing

I thank Prof. Dr.-Ing. Adolf J. Schwab very much for his encouraging comments and for his detailed critique of certain points in my book. I hope that my elaboration on points in his critique will help readers in their efforts to understand thoroughly some finer points of the mathematics involved and especially the corresponding English terminology.

(end of the author's response to Prof. Dr.-Ing. Adolf J. Schwab's comments)

4. The term “graph” in mathematics

In the book the terms “graphical notation” and “graph” are introduced in section 3.4.6. In mathematics the term “graph” is also used for something quite different, a structure in the sense of section 4.1.

The structure “graph” is defined in mathematics to consist of a set of *nodes* and a set of *edges* between nodes. An edge may or may not have a direction, i.e. it may represent either a connection from one node to another or it may simply connect two nodes. Nodes are also often called vertices. Values may be associated with nodes and/or with edges.

One of the many applications of this type of graph is to represent a finite state machine (defined in section 4.1.7 of the book). Each state is represented by a node and each transition between states is represented by an edge. Each edge is directed from the previous state to the next state. The name of each state is associated with the corresponding node. Associated with each edge are the corresponding input and output elements.

In the case of the example in section 2.12 of the book, the value associated with each state would be the name of the state as given in the leftmost column in Table 2.12-1. The value associated with each edge would consist of the pair of input and output elements as given in the body of the table.

5. A new example: To make or not to make? To sell or not to sell?

This is a very interesting example of a short and simple text in natural language which most people find difficult to understand. Each of the two individual sentences is relatively easy to understand, but in combination, they are not easy to interpret completely.

I became aware of this text some time after my book *The Language of Mathematics: Utilizing Math in Practice* had already been published. Had I been aware of it before publication, I certainly would have included it in the book.

Proposition: There are sentences in your native language that you can understand *only* if you are able to translate them into the Language of Mathematics.

Original statement in Italian:
1. Io non faccio quello che vendo.
2. Vendo tutto quello che faccio!

by Vasco Rossi, born in 1952, Italian singer and song writer. From the *Harenberg Sprachkalendar Italienisch 2013* for Martedi, 30 Aprile 2013.

English translation:
1. I do not make that which I sell.
2. I sell everything that I make!

Questions:
1. What does the first sentence mean? the second sentence?
2. What do they together imply?
3. Do I make anything?
4. Do I sell anything?
5. Most simply expressed, what do the two sentences together mean? What is the relationship between what I make and what I sell?

Values, Variables and Functions:
Note firstly that the verbs in both sentences are used in a stative sense. No clause describes a particular action that took place in the past, that is taking place in the present, or that will take place in the future. Instead, each clause specifies a characteristic habitual activity of the

subject "I". For example, "I sell …" here means "I am a seller of …" and "I make …" means "I am a maker of …". Each clause can, therefore, be modelled mathematically by a Boolean function. Two are needed: IMake(…) and ISell(…), where the argument represents the thing possibly made or sold.

To identify non-Boolean variables, look for nouns or pronouns. "I" appears only as the subject of each clause in each sentence, so the value "I" can be used directly if needed. No variable for "I" is needed.

In each sentence, two noun/pronouns are present. Each refers to the same thing and is a cross reference between the two clauses in each sentence. The thing or things being made or sold (or not made or not sold) are not stated specifically. Because the sentences can refer to any number of different things, we will use "x" as the name of a variable whose value is a thing being made or sold (or not). The set of all such possible things will be called "$\mathbb{Things}$".

Interpretation of the Values, Variables and Functions:

x: A variable whose value is a thing possibly being made or sold
IMake(x): "I make x"
ISell(x): "I sell x"
$\mathbb{Things}$: the set of things that can be referred to in the two sentences

Translation of the clauses and the sentences:

Consider first the clause "I do not make that". From the interpretation defined above the obvious translation is the negation of "I make that", i.e. "¬IMake(x)". The other clause in that sentence is "which I sell", or in subject-verb-object order, "I sell which", this "which" being the same thing as the "that" in the first clause. From the interpretation above the translation is "ISell(x)".

Therefore, the first sentence,

I do not make that which I sell.

will be translated by an appropriate combination of "¬IMake(x)" and "ISell(x)". The question is how to combine these translations of the two clauses so that the meaning of the English sentence is captured in the mathematical translation. The inherent meaning conveyed by such an English sentence appears to be an implication (if … then …), but the question remains in which direction the implication applies. Many people would say that the sentence means the same as

If I sell x, then I do not make x. [ISell(x) ⇒ ¬IMake(x)]

but some would not be sure. In principle the conjunctions ∧ and ∨ are also conceivably possible, as is equality (= or ⇔). Helpful in determining the appropriate connective in this case would be to rephrase the English sentence in the several corresponding ways and then to decide which alternative is the one actually meant. The original sentence and its possible rephrased versions are:

I do not make that which I sell.	
If I do not make x, then I sell x.	[¬IMake(x) ⇒ ISell(x)]
If I sell x, then I do not make x.	[ISell(x) ⇒ ¬IMake(x)]
I do not make x but (and) I sell x.	[¬IMake(x) ∧ ISell(x)]
I do not make x or I sell x.	[¬IMake(x) ∨ ISell(x)]
I do not make x is the same as I sell x.	[¬IMake(x) = ISell(x)]
I do not make x if and only if I sell x.	[¬IMake(x) ⇔ ISell(x)]

Here we assume that the team members have agreed upon

If I sell x, then I do not make x. [ISell(x) $\Rightarrow \neg$IMake(x)]

as the intended meaning of the first sentence "I do not make that which I sell".

Similar considerations applied to sentence 2 would lead many people to say that sentence 2, "I sell everything that I make", means the same as

If I make x, then I sell x. [IMake(x) $\Rightarrow$ ISell(x)]

but with some doubt as in the case of sentence 1 above.

The logical mathematical expressions for the two sentences are, as usual, combined with the logical function $\wedge$ (and).

If one cannot decide which of the above alternatives captures exactly the meaning of the original English sentence, then one can construct a table showing all combinations of making, not making, selling and not selling a thing x. Then one should decide whether or not each such combination is consistent with, i.e. satisfies, the original sentence. When examining each combination for consistency with the sentence, it is sometimes easier and the answer is clearer if one asks whether or not the combination in question violates or contradicts the sentence in question. If the combination in question does not violate or contradict the sentence in question, then the combination is consistent with the sentence.

The table below shows all combinations of making, not making, selling and not selling a thing x. For each combination, its consistency with sentence 1 and with sentence 2 is shown.

	I make x	I sell x	consistent with		
	IMake(x)	ISell(x)	sentence 1	sentence 2	both sentences
I do not make x and I do not sell x.	false	false	true	true	true
I do not make x and I sell x.	false	true	true	true	true
I make x and I do not sell x.	true	false	true	false	false
I make x and I sell x.	true	true	false	true	false

Each of the columns for sentence 1 and for sentence 2 contains only one entry with the value false. This is characteristic of the logical implication function (see Table 3.3-1 in the book). The logical expression corresponding to sentence 1 is, therefore

ISell(x) $\Rightarrow \neg$IMake(x) or, equivalently, IMake(x) $\Rightarrow \neg$ISell(x) [sentence 1]

and the logical expression for sentence 2 is

IMake(x) $\Rightarrow$ ISell(x) or, equivalently, $\neg$ISell(x) $\Rightarrow \neg$IMake(x) [sentence 2]

Formally, the expression for sentence 1 applies to all possible values of x, that is, it should be quantified over all x, and correspondingly for sentence 2.

Semi-formal derivation of the solution:
From the table above it can be seen that the only combinations of making, not making, selling and not selling that are consistent with both sentences are

I do not make x and I do not sell x.

I do not make x and I sell x.

Thus, the only situations consistent with both sentences require that "I do not make x" for all x. Regarding selling, both possibilities are left open; I may or may not sell x. I.e., the two sentences together require that I do not make anything, but say nothing about what, if anything, I sell.

This same conclusion follows from an algebraic simplification of the corresponding mathematical expression. Sentences 1 and 2 together can be expressed in the Language of Mathematics as

$[\mathrm{ISell}(x) \Rightarrow \neg\mathrm{IMake}(x)] \wedge [\mathrm{IMake}(x) \Rightarrow \mathrm{ISell}(x)]$

which can be simplified as follows:

$[\mathrm{ISell}(x) \Rightarrow \neg\mathrm{IMake}(x)] \wedge [\mathrm{IMake}(x) \Rightarrow \mathrm{ISell}(x)]$

= [section 5.2.3, Lemma 4]

$[\neg\mathrm{ISell}(x) \vee \neg\mathrm{IMake}(x)] \wedge [\neg\mathrm{IMake}(x) \vee \mathrm{ISell}(x)]$

= [Table 5.1.2 in the errata, last line]

$[\neg\mathrm{ISell}(x) \wedge \neg\mathrm{IMake}(x)] \vee [\neg\mathrm{ISell}(x) \wedge \mathrm{ISell}(x)] \vee$
$[\neg\mathrm{IMake}(x) \wedge \neg\mathrm{IMake}(x)] \vee [\neg\mathrm{IMake}(x) \wedge \mathrm{ISell}(x)]$

= [identities 5.2.4-1, 5.2.4-3 and 5.2.4-6]

$[\neg\mathrm{ISell}(x) \wedge \neg\mathrm{IMake}(x)] \vee [\neg\mathrm{IMake}(x)] \vee [\neg\mathrm{IMake}(x) \wedge \mathrm{ISell}(x)]$

= [identity 5.2.4-10]

$[\neg\mathrm{IMake}(x)]$

I.e., translated back into English, the two sentences together say that "I do not make x", for all x, i.e. "I do not make anything", no more, no less. They say nothing about my selling anything.

The mathematical model:

$(\mathrm{IMake}: \mathbb{Things} \to \mathbb{B}) \wedge (\mathrm{ISell}: \mathbb{Things} \to \mathbb{B})$ [header, see section 6.13 of the book]
$\wedge [\wedge\ x : x \in \mathbb{Things} : \mathrm{ISell}(x) \Rightarrow \neg\mathrm{IMake}(x)]$ [sentence 1]
$\wedge [\wedge\ x : x \in \mathbb{Things} : \mathrm{IMake}(x) \Rightarrow \mathrm{ISell}(x)]$ [sentence 2]

The mathematical model above implies that

$[\wedge\ x : x \in \mathbb{Things} : \neg\mathrm{IMake}(x)]$

The proof follows the pattern of the simplification in the section "**Semi-formal derivation of the solution**" above.

Answers to the questions:
1. If I sell x then I do not make x. If I make x then I sell x.
2. I do not make anything. I may or may not sell anything in particular.
3. No, I do not make anything.
4. Perhaps, perhaps not. It is consistent with the two sentences that I sell nothing or that I sell one or more things.

5. I do not make anything. Together, the two sentences say absolutely nothing whatsoever about what I sell or even whether or not I sell anything.

A short alternative derivation of the solution:
Begin by reformulating sentences 1 and 2 into the form if … then … (implications):

English	mathematical notation	
If I sell x, then I do not make x.	ISell(x) ⇒ ¬IMake(x)	[sentence 1']
If I make x, then I sell x.	IMake(x) ⇒ ISell(x)	[sentence 2']

Then reformulate sentence 2' by negating each clause and reversing the order of implication. By a fundamental theorem of logic, the result is equivalent to sentence 2' and hence also to sentence 2:

If I do not sell x, then I do not make x.	¬ISell(x) ⇒ ¬IMake(x)	[sentence 2"]

The sentences 1 and 2 are then equivalent to the sentences 1' and 2":

If I sell x, then I do not make x.	ISell(x) ⇒ ¬IMake(x)	[sentence 1']
If I do not sell x, then I do not make x.	¬ISell(x) ⇒ ¬IMake(x)	[sentence 2"]

In other words, I do not make x, regardless of whether or not I sell x, and that for all x. That is, I do not make anything. The combination of sentences says nothing about whether or not I sell any particular x.

6. Shopping mall door controller: a "monster"?

In a review of this book, a reader referred to the model of a shopping mall door controller in section 8.13 as a "50 page monster".

While I would not use the word "monster" to describe the model, it certainly is long, large and extensive and contains a great amount of detail. This is typical of many "real-world" applications of mathematics to large systems, including controllers of many types. While no one part of the logic is particularly complicated, in total, the system's logic can be complex and difficult to understand.

A mathematical approach is the only way to come to grips properly with a problem requiring such extensive and complex logic.

If you think that the shopping mall controller is a "monster", consider the control logic for a system of n elevators moving people from one floor to another in a building with m floors. If you do try, I would suggest starting with only one elevator in order to get a feeling for the task in general and for such details as when and where to stop the elevator, when to ignore a new request temporarily (e.g. because too little time is available to stop the elevator), etc. Then extend your design to 2 elevators, but in a way not dependent on the number 2, i.e. in such a way that your design can be extended to more elevators.

The following general guidelines contribute to a simple and systematically structured design:

- Subdivide the system (e.g. the controller) into several subsystems (subcontrollers), e.g. one for each elevator, one for the floor requests of people in each elevator, one for the floor and directional requests of people on each floor waiting for an elevator, etc. The interfaces between the subsystems should require little data to be exchanged. The lock controller in section 7.5.2 of the book is an example in which the overall controller system is subdivided into the "master" and "subsidiary" controllers.
- Define the states of the system and of each subsystem thoughtfully and carefully. Getting this part of the design right is the most important step in the entire process.

Do not try to do this too quickly. Do not take a superficial approach to this task. Time spent on this step pays off well later. A poorly thought out definition of the system's states will lead to great difficulty and a logical mess when trying to define the various state transitions and the conditions for them. Especially in this phase should one apply the KISSS principal: Keep It Simple and Systematically Structured.

- List every combination of input values and states of the controller and of its several subcontrollers. List not only the "normal" combinations that are expected to arise in operation, but *all logically conceivable combinations*. Failures and errors in actual operation can give rise to unplanned and unexpected inputs and states, and it is important to consider these possibilities in the original design, even if it appears "impossible" that they can arise.

If your initial design for a realistic system involving a number of different components and a variety of different kinds of inputs does not appear to be a "monster", you should seriously ask yourself if you have overlooked something. Even a controller for a single door on a train, for example, is by no means trivial, especially when all safety requirements are considered.